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# Oscillation of Solutions of Impulsive Nonlinear Parabolic Differential-Difference Equations

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Sufficient conditions for oscillation of the solutions of impulsive nonlinear parabolic differential-difference equations are obtained.

## 1. INTRODUCTION

During the period 1970–1980 many results in the oscillation theory for partial differential equations (PDE) were obtained. After 1980 the investigations in this theory were extended to new classes of equations: PDE with deviating arguments and partial integrodifferential equations (Yoshida, 1979, 1985, 1986, 1987, 1992a,b).

The theory of impulsive PDE is a new branch of the theory of PDE. Impulsive PDE have been objects of intensive investigation since 1991 and many papers have been devoted to the fundamental and qualitative theory for impulsive PDE, impulsive population dynamics, and numerical methods for impulsive PDE (Ahmad and Rama Mohana Rao, n.d.; Bainov *et al.*, 1994, 1995a–c; Byszewski, 1992, 1993; Gupta, 1994).

In the present paper sufficient conditions are obtained such that every solution of impulsive nonlinear parabolic differential-difference equations satisfying certain boundary conditions is oscillating.

## 2. PRELIMINARY NOTES

Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are given

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numbers and $t_{k+l} = t_k + \sigma$, $k = 0, 1, \ldots$, where $\sigma = \text{const} > 0$, and $l$ is a fixed natural number.

Define $J_{\text{imp}} = \{t_k\}_{k=1}^{\infty}$, $R_+ = [0, +\infty)$, $E^0 = [-\sigma, 0] \times \overline{\Omega}$, $E = (0, +\infty) \times \Omega$, $E^* = R_+ \times \overline{\Omega}$, $E_{\text{imp}} = \{(t, x) \in E: t \in J_{\text{imp}}\}$, $E^*_{\text{imp}} = \{(t, x) \in E^*: t \in J_{\text{imp}}\}$.

Let $C_{\text{imp}}[E^0 \cup E^*, R]$ be the class of all functions $u: E^0 \cup E^* \to R$ such that:

(i) The restriction of $u$ to the set $E^0 \cup E^* \backslash E^*_{\text{imp}}$ is a continuous function.
(ii) For each $(t, x) \in E^*_{\text{imp}}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u(q, s) = u(t^-, x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q>t}} u(q, s) = u(t^+, x)$$

and $u(t, x) = u(t^+, x)$ for $(t, x) \in E^*_{\text{imp}}$.

The class of functions $C_{\text{imp}}[E^*, R]$ is defined analogously, with $E^*$ written instead of $E^0 \cup E^*$ in the above definition.

Consider the nonlinear parabolic differential-difference equation

$$u_t(t, x) - a(t)\Delta u(t, x) + p(t, x)f(u(t - \sigma, x)) = 0,$$
$$(t, x) \in E\backslash E_{\text{imp}} \tag{1}$$

subject to the impulsive condition

$$u(t, x) - u(t^-, x) = g(t, x, u(t^-, x)), \qquad (t, x) \in E^*_{\text{imp}} \tag{2}$$

and the boundary conditions

$$\frac{\partial u}{\partial n}(t, x) + \gamma(t, x)u(t, x) = 0, \qquad (t, x) \in (R_+\backslash J_{\text{imp}}) \times \partial\Omega \tag{3}$$

or

$$u(t, x) = 0, \qquad (t, x) \in (R_+\backslash J_{\text{imp}}) \times \partial\Omega \tag{4}$$

The functions $a: R_+ \to R$, $p: E^* \to R$, $f: R \to R$, $g: E^*_{\text{imp}} \times R \to R$, $\gamma: R_+ \times \partial\Omega \to R$ are given.

*Definition 1.* The function $u: E^0 \cup E^* \to R$ is called a solution of the problem (1)–(3) [(1), (2), (4)] if:

(i) $u \in C_{\text{imp}}[E^0 \cup E^*, R]$, there exist the derivatives $u_t(t, x)$, $u_{x_i x_i}(t, x)$, $i = 1, \ldots, n$, for $(t, x) \in E\backslash E_{\text{imp}}$ and $u$ satisfies (1) on $E\backslash E_{\text{imp}}$.
(ii) $u$ satisfies (2), (3) [(2), (4)].

*Definition 2.* The nonzero solution $u(t, x)$ of equation (1) is said to be nonoscillating if there exists a number $\mu \geq 0$ such that $u(t, x)$ has a constant sign for $(t, x) \in [\mu, +\infty) \times \Omega$. Otherwise, the solution is said to oscillate.

For the function sign we adopt the following definition:

$$\operatorname{sign} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Introduce the following assumptions:

**H1.** $a \in C_{\text{imp}}[R_+, R_+]$.
**H2.** $p \in C_{\text{imp}}[E^*, R_+]$.
**H3.** $g \in C(E^*_{\text{imp}} \times R, R)$.
**H4.** $\gamma \in C_{\text{imp}}[R_+ \times \partial\Omega, R_+]$.
**H5.** $f \in C(R, R), f(u) = -f(-u)$ for $u \geq 0$, $f$ is a positive and convex function in the interval $(0, +\infty)$.

In the sequel the following notations will be used:

$$P(t) = \min\{p(t, x): x \in \overline{\Omega}\}$$

$$V(t) = \int_\Omega u(t, x)\,dx\left(\int_\Omega dx\right)^{-1}$$

## 3. MAIN RESULTS

We give sufficient conditions for the oscillation of the solutions of problem (1)–(3).

*Lemma 1.* Let the following conditions hold:

1. Assumptions H1–H5 are fulfilled.
2. $u \in C^2(E\backslash E_{\text{imp}}) \cap C^1(E^*\backslash E^*_{\text{imp}})$ is a positive solution of the problem (1)–(3) in the domain $E$.
3. $g(t_k, x, \xi) \leq L_k\xi$, $k = 1, 2, \ldots, x \in \overline{\Omega}$, $\xi \in R_+$, $L_k \geq 0$ are constants.

Then the function $V(t)$ satisfies for $t \geq \sigma$ the impulsive differential inequality

$$V'(t) + P(t)f(V(t-\sigma)) \leq 0, \qquad t \neq t_k \tag{5}$$

$$V(t_k) \leq (1 + L_k)V(t_k^-) \tag{6}$$

*Proof.* Let $t \geq \sigma$. Integrating equation (1) with respect to $x$ over the domain $\Omega$, we obtain

$$\frac{d}{dt}\int_\Omega u(t, x)\,dx - a(t)\int_\Omega \Delta u(t, x)\,dx$$
$$+ \int_\Omega p(t, x)f(u(t-\sigma, x))\,dx = 0, \qquad t \neq t_k \tag{7}$$

From the Green formula and H4 it follows that

$$\int_{\Omega} \Delta u(t, x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = -\int_{\partial\Omega} \gamma(t, x) u(t, x)\, dS \le 0, \qquad t \ne t_k \quad (8)$$

Moreover, for $t \ne t_k$, the Jensen inequality enables us to get

$$\begin{aligned} &\int_{\Omega} p(t, x) f(u(t - \sigma, x))\, dx \\ &\quad \ge P(t) \int_{\Omega} f(u(t - \sigma, x))\, dx \\ &\quad \ge P(t) f\left(\int_{\Omega} u(t - \sigma, x)\, dx \left(\int_{\Omega} dx\right)^{-1}\right) \int_{\Omega} dx = P(t) f(V(t - \sigma)) \int_{\Omega} dx \quad (9) \end{aligned}$$

In virtue of (8) and (9) we obtain from (7) that

$$V'(t) + P(t) f(V(t - \sigma)) \le 0, \qquad t \ne t_k$$

For $t = t_k$ we have that

$$V(t_k) - V(t_k^-) \le L_k \left(\int_{\Omega} dx\right)^{-1} \int_{\Omega} u(t_k^-, x)\, dx = L_k V(t_k^-)$$

that is,

$$V(t_k) \le (1 + L_k) V(t_k^-) \quad \blacksquare$$

*Definition 3.* The solution

$$V \in C_{\text{imp}}[[-\sigma, 0] \cup \mathbb{R}_+, \mathbb{R}] \cap C^1\left(\bigcup_{k=0}^{\infty} (t_k, t_{k+1}), \mathbb{R}\right)$$

of the differential inequality (5), (6) is called eventually positive (negative) if there exists a number $t^* \ge 0$ such that $V(t) > 0$ $[V(t) < 0]$ for $t \ge t^*$.

*Theorem 1.* Let the following conditions hold:

1. Assumptions H1–H5 are fulfilled.
2. $g(t_k, x, \xi) \le L_k \xi$, $k = 1, 2, \ldots$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \ge 0$ are constants and $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.
3. Each eventually positive solution of the differential inequality (5), (6) tends to zero as $t \to \infty$.

Then each nonzero solution $u \in C^2(E \backslash E_{\text{imp}}) \cap C^1(E^* \backslash E^*_{\text{imp}})$ of problem (1)–(3) either oscillates in the domain $E$, or

$$\lim_{t\to\infty} \int_\Omega u(t, x)\, dx = 0$$

*Proof.* Suppose the conclusion of the theorem is not true, i.e., $u(t, x)$ is a nonzero solution of the problem (1)–(3) which is of the class $C^2(E \backslash E_{\text{imp}}) \cap C^1(E^* \backslash E^*_{\text{imp}})$, it has a constant sign in the domain $E_\mu = [\mu, +\infty) \times \Omega$, $\mu \geq 0$, and

$$\int_\Omega u(t, x)\, dx \not\to 0 \qquad \text{as} \quad t \to \infty$$

Without loss of generality we may assume that $u(t, x) > 0$ for $(t, x) \in E_\mu$. Then it follows from Lemma 1 that the function $V(t)$ is a positive solution of the differential inequality (5), (6) for $t \geq \mu + \sigma$ and $V(t) \not\to 0$ as $t \to \infty$, which contradicts condition 3 of the theorem. ∎

*Theorem 2.* Let the following conditions hold:

1. $P \in C_{\text{imp}}[\mathbb{R}_+, \mathbb{R}_+]$, $\int_{t^*}^\infty P(\tau)\, d\tau = +\infty$ for each $t^* \geq 0$.
2. $\sum_{k=1}^\infty L_k < +\infty$, $L_k \geq 0$, $k = 1, 2, \ldots$, are constants.
3. $f(u) \geq Mu$, $u \geq 0$, $M > 0$ is a constant.

Then each eventually positive solution of the differential inequality (5), (6) tends to zero as $t \to \infty$.

*Proof.* Let $V(t)$ be an eventually positive solution of the differential inequality (5), (6), that is, there exists a point $t^* \geq 0$ such that $V(t) > 0$ for $t \geq t^*$. Then for $t \geq t^* + \sigma$

$$V'(t) + MP(t)V(t - \sigma) \leq 0, \qquad t \neq t_k$$
$$V(t_k) \leq (1 + L_k)V(t_k^-)$$

Since $V'(t) \leq 0$ for $t \geq t^* + \sigma$, $t \neq t_k$, we obtain for each $\bar{t}_1 \geq t^* + \sigma$ that

$$V(t) \leq \prod_{\bar{t}_1 < t_k \leq t} (1 + L_k)V(\bar{t}_1)$$

Thus for $t \geq t^* + 2\sigma$ we get the estimate

$$V(t) \leq \prod_{t-\sigma < t_k \leq t} (1 + L_k)V(t - \sigma)$$

and consequently,

$$V'(t) + \frac{MP(t)}{\prod_{t-\sigma < t_k \leq t} (1 + L_k)} V(t) \leq 0$$

for $t \neq t_k$, $t \geq t^* + 2\sigma$. Direct calculation gives us

$$V(t) \leq V(t^* + 2\sigma) \prod_{t^*+2\sigma<t_k\leq t} (1 + L_k)\exp\left(-M \int_{t^*+2\sigma}^{t} \frac{P(\tau)}{\prod_{\tau-\sigma<t_k\leq\tau} (1 + L_k)}\, d\tau\right) \tag{10}$$

The estimate (10) and the fact that $V(t)$ is eventually positive imply that $\lim_{t\to\infty} V(t) = 0$. ∎

*Corollary 1.* Let the following conditions hold:

1. Assumptions H1–H5 are fulfilled.
2. $g(t_k, x, \xi) \leq L_k\xi$, $k = 1, 2, \ldots, x \in \overline{\Omega}$, $\xi \in R_+$, $L_k \geq 0$ are constants such that $\sum_{k=1}^{\infty} L_k < +\infty$ and $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.
3. $\int_{t^*}^{\infty} P(\tau)\, d\tau = +\infty$ for each $t^* \geq 0$.
4. $f(u) \geq Mu$ for $u \geq 0$, $M > 0$ is a constant.

Then each nonzero solution $u \in C^2(E\backslash E_{\text{imp}}) \cap C^1(E^*\backslash E^*_{\text{imp}})$ of problem (1)–(3) either oscillates in the domain $E$, or

$$\lim_{t\to\infty} \int_{\Omega} u(t, x)\, dx = 0$$

Corollary 1 follows from Theorems 1 and 2.

Now we give sufficient conditions for oscillation of the solutions of the problem (1), (2), (4).

Consider the following Dirichlet problem:

$$\begin{aligned} \Delta\varphi + \alpha\varphi &= 0 \quad \text{in } \Omega \\ \varphi|_{\partial\Omega} &= 0 \end{aligned} \tag{11}$$

where $\alpha = \text{const}$. It is known that the smallest eigenvalue $\alpha_0$ of the problem (11) is positive and the corresponding eigenfunction $\varphi_0(x) > 0$ for $x \in \Omega$. Without loss of generality we may assume that $\varphi_0$ is normalized, i.e., $\int_{\Omega} \varphi_0(x)\, dx = 1$.

Introduce the notation

$$W(t) = \int_{\Omega} u(t, x)\varphi_0(x)\, dx$$

*Lemma 2.* Let the following conditions hold:

1. Assumptions H1–H3, H5 are fulfilled.
2. $u \in C^2(E\backslash E_{\text{imp}}) \cap C^1(E^*\backslash E^*_{\text{imp}})$ is a positive solution of the problem (1), (2), (4) in the domain $E$.

3. $g(t_k, x, \xi) \le L_k\xi$, $k = 1, 2, \ldots, x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \ge 0$ are constants.

Then the function $W(t)$ satisfies for $t \ge \sigma$ the impulsive differential inequality

$$W'(t) + \alpha_0 a(t)W(t) + P(t)f(W(t-\sigma)) \le 0, \qquad t \ne t_k \tag{12}$$

$$W(t_k) \le (1 + L_k)W(t_k^-) \tag{13}$$

*Proof.* Let $t \ge \sigma$. We multiply the both sides of equation (1) by the eigenfunction $\varphi_0(x)$, and integrating with respect to $x$ over $\Omega$, we obtain

$$\frac{d}{dt}\int_\Omega u(t,x)\varphi_0(x)\,dx - a(t)\int_\Omega \Delta u(t,x)\varphi_0(x)\,dx$$
$$+ \int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)\,dx = 0, \qquad t \ne t_k \tag{14}$$

From the Green formula it follows that

$$\int_\Omega \Delta u(t,x)\varphi_0(x)\,dx = \int_\Omega u(t,x)\Delta\varphi_0(x)\,dx$$
$$= -\alpha_0\int_\Omega u(t,x)\varphi_0(x)\,dx = -\alpha_0 W(t), \qquad t \ne t_k \tag{15}$$

where $\alpha_0 > 0$ is the smallest eigenvalue of the problem (11).

Moreover, from the Jensen inequality, we have

$$\int_\Omega p(t,x)f(u(t-\sigma,x))\varphi_0(x)\,dx$$
$$\ge P(t)\int_\Omega f(u(t-\sigma,x))\varphi_0(x)\,dx$$
$$\ge P(t)f\left(\int_\Omega u(t-\sigma,x)\varphi_0(x)\,dx\right) = P(t)f(W(t-\sigma)), \qquad t \ne t_k \tag{16}$$

Making use of (15) and (16), we obtain from (14) that

$$W'(t) + \alpha_0 a(t)W(t) + P(t)f(W(t-\sigma)) \le 0, \qquad t \ne t_k$$

For $t = t_k$ we have that

$$W(t_k) - W(t_k^-) \le L_k\int_\Omega u(t_k^-, x)\varphi_0(x)\,dx = L_k W(t_k^-)$$

that is,

$$W(t_k) \leq (1 + L_k)W(t_k^-) \quad \blacksquare$$

Analogously to Theorem 1, we can prove the following theorem.

*Theorem 3.* Let the following conditions hold:

1. Assumptions H1–H3, H5 are fulfilled.
2. $g(t_k, x, \xi) \leq L_k\xi$, $k = 1, 2, \ldots, x \in \overline{\Omega}$, $\xi \in \mathbb{R}_+$, $L_k \geq 0$ are constants and $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.
3. Each eventually positive solution of the differential inequality (12), (13) tends to zero as $t \to \infty$.

Then each nonzero solution $u \in C^2(E\backslash E_{\text{imp}}) \cap C^1(E^*\backslash E^*_{\text{imp}})$ of problem (1), (2), (4) either oscillates in the domain $E$ or

$$\lim_{t\to\infty} \int_\Omega u(t, x)\varphi_0(x)\,dx = 0$$

*Theorem 4.* Let the following conditions hold:

1. Assumption H1 is satisfied.
2. The conditions of Theorem 2 hold.

Then each eventually positive solution of the differential inequality (12), (13) tends to zero as $t \to \infty$.

The proof of Theorem 4 is analogous to the proof of Theorem 2. It is omitted here.

*Corollary* 2. Let the conditions of Corollary 1 be fulfilled except for H4.
Then each nonzero solution $u \in C^2(E\backslash E_{\text{imp}}) \cap C^1(E^*\backslash E^*_{\text{imp}})$ of problem (1), (2), (4) either oscillates in the domain $E$, or

$$\lim_{t\to\infty} \int_\Omega u(t, x)\varphi_0(x)\,dx = 0$$

Corollary 2 follows from Theorems 3 and 4.

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